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A GLOBAL ŁOJASIEWICZ INEQUALITY FOR ALGEBRAIC VARIETIES

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ABSTRACT. Let X be the locus of common zeros of polynomials $f_1, \dots, f_k$ in n complex variables. A global upper bound for the distance to X is given in the form of a Łojasiewicz inequality. The exponent in this inequality is bounded by $d^{\min(n,k)}$ where $d = \max(3, \deg f_i)$. The estimates are also valid over an algebraically closed field of any characteristic.

Let f be a real analytic function on $\mathbb{R}^n$ and let $Z = \{x \in \mathbb{R}^n | f(x) = 0\}$. Let $\operatorname{dist}(x, Z) = \inf_{z\in Z} ||x - z||$ where $||\cdot||$ denotes the Euclidean norm. For any $x \in \mathbb{R}^n$ one expects to be able to compare $\operatorname{dist}(x, Z)$ and $f(x)$. This is done by the Łojasiewicz inequality:

1. **Theorem** (Łojasiewicz [L1, Theorem 17], see also [M, Theorem 4.1]). *With the above notation, for any compact set K there are positive constants C and α such that* $\operatorname{dist}(x, Z)^\alpha \le C \cdot |f(x)|$ *for every $x \in K$.*

In general α can be large. For example [L2, p. 85] if $f(x, y) = y^{2m} + (y - x^m)^2$ then $\alpha \ge 2m^2$. Also, it is not clear how α depends on f.

If $Z \subset \mathbb{C}^n$ is defined by complex analytic equations $f_1 = \cdots = f_k = 0$, then viewed as a real analytic set $Z \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$ it is defined by $f = 0$ where $f = |f_1|^2 + \cdots + |f_k|^2$ is a real analytic function. Thus the Łojasiewicz inequality applies to complex analytic or algebraic sets too.

If the defining equations f_i are polynomials, one would like to estimate the exponent α in terms of the degrees of the polynomials. Recently Brownawell [B1] (see also [BY, Section 3]) proved such a bound for polynomials over the complex field. A polynomial has more complex zeros than real ones; thus one expects the complex case to be easier. In fact Brownawell's methods (and also ours) do not apply in the real case.

2. **Theorem** (Brownawell [B1]). *Let $f_1, \dots, f_k \in \mathbb{C}[z_1, \dots, z_n]$ and $D = \max \deg f_i$. Let $Z = \{z \in \mathbb{C}^n | f_1(z) = \cdots = f_k(z) = 0\}$. Then there is a constant $C > 0$ such that*

$$\left(\frac{\min(\operatorname{dist}(z, Z), 1)}{1 + ||z||^2}\right)^{(n+1)^2 D^{\min(n,k)}} \le C \cdot \max_i |f_i(z)|,$$

where $||z||^2 = |z_1|^2 + \cdots + |z_n|^2$.

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The aim of this paper is to find the best possible exponent in terms of the degrees of the polynomials f_i. We need the following notation:

3. *Notation.* Given natural numbers $n \geq 2$ and $d_1 \geq \cdots \geq d_k$ let

$$B(n, d_1, \dots, d_k) = \begin{cases} d_1 \cdot \dots \cdot d_k & \text{if } k \leq n; \\ d_1 \cdot \dots \cdot d_{n-1} \cdot d_k & \text{if } k > n. \end{cases}$$

For technical reasons related to the proofs in [B2] and [K] we also define

$$\overline{B}(n, d_1, \dots, d_k) = \left(\frac{3}{2}\right)^j B(n, d_1, \dots, d_k) + \theta,$$

where $j = \#\{i < \min(k, n) - 1 | d_i = 2\}$ and

$$\theta = \begin{cases} 1 & \text{if } k > n \text{ and } d_{n-1} = 2, \\ 0 & \text{otherwise.} \end{cases}$$

We extend the above notation to any sequence $d_1, \dots, d_k$ by first ordering it and then applying the above definitions.

4. **Definition.** Let K be an algebraically closed field. By an absolute value we mean a valuation $|\ | : K \to [0, \infty)$ which satisfies the triangle inequality (and which can be Archimedean or not). Any basis of K^n leads to a norm

$$||(x_1, \dots, x_n)|| = \sqrt{|x_1|^2 + \cdots + |x_n|^2}.$$

If $V \subset K^n$ then we define

$$\operatorname{dist}(x, V) = \inf_{y \in V} ||x - y||.$$

The following is our main result:

5. **Theorem** (Łojasiewicz-type inequality). *Let K be an algebraically closed field (any characteristic) and let $|\ |$ be an absolute value as in* (4). *Let $f_1, \dots, f_k \in K[x_1, \dots, x_n]$ be polynomials and let $d_i = \deg f_i$. Assume that $n \geq 2$. Let $V = V(f_1, \dots, f_k) \subset K^n$ be the common zero set of these polynomials. Assume that V is nonempty.*

Then there is a positive integer $m \leq \overline{B}(n, d_1, \dots, d_k)$ and a constant $C > 0$ (both depending on the f_i) such that

$$\operatorname{dist}(x, V)^m \leq C \cdot \max_i\{|f_i(x)|\} \cdot (1 + ||x||)^{\overline{B}(n, d_1, \dots, d_k)}$$

holds for all $x \in K^n$.

Since $\operatorname{dist}(z, Z) \leq C' \cdot (1 + ||z||)$ holds for some $C' > 0$, (5) implies the following improvement of Brownawell's result:

6. **Corollary.** *Let $f_1, \dots, f_k \in \mathbb{C}[z_1, \dots, z_n]$ and let $d_i = \deg f_i$. Let $Z \subset \mathbb{C}^n$ be the common zero set of these polynomials. Then there is a constant $C > 0$ such that*

$$\left(\frac{\min(\operatorname{dist}(z, Z), 1)}{1 + ||z||}\right)^{\overline{B}(n, d_1, \dots, d_k)} \leq C \cdot \max_i |f_i(z)| \quad \text{and} \tag{6.1}$$

$$\left(\frac{\operatorname{dist}(z, Z)}{1 + ||z||^2}\right)^{\overline{B}(n, d_1, \dots, d_k)} \leq C \cdot \max_i |f_i(z)| \tag{6.2}$$

holds for all $z \in \mathbb{C}^n$. □

The proof of (5) will rest on Brownawell's version [B2] of the effective Nullstellensatz [K]:

7. Theorem [B2, Main Proposition]. *Let K be a field and let $\overline{f}_1, \dots, \overline{f}_k \in K[x_0, \dots, x_n]$ be homogeneous polynomials of degree $d_1, \dots, d_k$ respectively. Assume that $n \geq 2$.*

Then there are prime ideals $P_1, \dots, P_s$ containing $(\overline{f}_1, \dots, \overline{f}_k)$ and there are natural numbers $e_1, \dots, e_s$ such that

$$\prod_{i=1}^{s} P_i^{e_i} \subset (\overline{f}_1, \dots, \overline{f}_k) \qquad \textit{and}$$

$$\sum_{i=1}^{s} e_i \cdot \deg P_i \leq \overline{B}(n, d_1, \dots, d_k).$$

The "analytic" part of the proof of (5) is based on the following lemma in which we use the notation of (4)

8. Lemma. *Let $Z \subset K^n$ be an irreducible subvariety of dimension k and degree d. Then there are finitely many polynomials g_i of degree at most d vanishing on Z and a constant C such that*

$$\operatorname{dist}(x, Z)^d \leq C \cdot \max_i\{|g_i(x)|\}.$$

Proof. Fix a generic projection $\Pi: K^n \to L$ where L is a k dimensional linear subspace. The restriction $\pi: Z \to L$ is finite of degree d and surjective. For any given $x \in K^n$ let

$$\operatorname{dist}_\Pi(x, Z) = \min_{y \in \pi^{-1}(\Pi(x))} ||x - y||.$$

It is clear that

$$\operatorname{dist}(x, Z) \leq \operatorname{dist}_\Pi(x, Z).$$

Therefore it is sufficient to prove that

$$\operatorname{dist}_\Pi(x, Z)^d \leq C \cdot \max_i\{|g_i(x)|\}. \tag{9}$$

If $\dim Z = n-1$ then Z is defined by a single polynomial $g \in K[x_1, \dots, x_n]$ of degree d. Since a different basis gives a norm which is bounded by constant multiples of the first norm from below and above, we may choose coordinates such that $\Pi(x_1, \dots, x_n) = (x_1, \dots, x_{n-1})$. Then $g = cx_n^d + \cdots$ where $c \neq 0$. Let $y_1, \dots, y_d \in Z$ be the preimages of $\Pi(x)$ under π (with multiplicities). Then

$$\operatorname{dist}(x, Z)^d \leq \prod_{i=1}^{d} \operatorname{dist}(x, y_i) = |c^{-1}g(x)| = |c^{-1}| \cdot |g(x)|.$$

Thus $C = |c^{-1}|$ is the desired constant.

We prove (9) by induction on the codimension. Suppose $\operatorname{codim} Z > 1$; fix a hyperplane $H \supset L$ and $d+1$ generic lines $A_i \subset \ker \Pi$. Let $p_i: K^n \to H$ be the projections with kernel A_i. Let $Z_i = p_i(Z) \subset H$ be the images. By assumption (9) is true for the Z_i.

It suffices to show that there is a constant C' such that

$$\mathrm{dist}_\Pi(x, Z) \le C' \cdot \max_i\{\mathrm{dist}_\Pi(p_i(x), Z_i)\}.$$

To see this define the ε-neighborhood of A_i as

$$U_\varepsilon(A_i) \stackrel{\mathrm{def}}{=} \{x \in K^n | x = x_a + x_h;\ x_a \in A_i;\, x_h \in H \text{ and } \|x_h\| < \varepsilon \cdot \|x_a\|\}.$$

Note that if $x \notin U_\varepsilon(A_i)$ then $\|x\| \le (1+\varepsilon^{-1})\|p_i(x)\|$. Now choose ε so that the ε-neighborhoods of the A_i do not intersect each other.

Let $\pi^{-1}(\Pi(x)) = \{y_1, \dots, y_p\}$ (as sets); $p \le d$. There are $(d+1)$ lines A_i and at most d points $x - y_i$. Thus there is an index j (depending on x) such that

$$\{x - y_1, \dots, x - y_p\} \cap U_\varepsilon(A_j) = \varnothing.$$

This implies that

$$\mathrm{dist}_\Pi(x, Z) \le (1+\varepsilon^{-1}) \max_j\{\mathrm{dist}_\Pi(p_j(x), Z_j)\}. \quad \square$$

10. *Remark.* It follows from the above proof that for a nonconstant polynomial $f \in K[x_1, \dots, x_n]$

$$\mathrm{dist}(x, V(f))^{\deg f} \le C \cdot |f(x)|.$$

Now we can prove (5).

We introduce a new variable x_0 and homogenize the polynomials $f_1, \dots, f_k$ to get $\overline{f}_1, \dots, \overline{f}_k$. Let $P_1, \dots, P_s$ be the prime ideals in (7). Assume that they are indexed such that $x_0 \in P_{r+1} \cap \cdots \cap P_s$ and $x_0 \notin P_1 \cup \cdots \cup P_r$. Let $Z_1, \dots, Z_r$ be the affine varieties in K^n corresponding to $P_1, \dots, P_r$. Then $V = Z_1 \cup \cdots \cup Z_r$.

By (8) for each Z_i we can find a finite collection of polynomials $\{g_{i,j}\}$ of degree at most $z_i := \deg Z_i = \deg P_i$ and a positive constant C_i such that g_{ij} vanishes on Z_i for each j and

$$\mathrm{dist}(x, Z_i)^{z_i} \le C_i \cdot \max_j\{|g_{i,j}(x)|\}.$$

Let $e_1, \dots, e_s$ be as in (7). Then

$$\begin{aligned}\mathrm{dist}(x, V)^{z_1e_1+\cdots+z_re_r} &\le \prod_{i=1}^r \mathrm{dist}(x, Z_i)^{z_ie_i} \\ &\le \prod_{i=1}^r C_i^{e_i} \prod_{i=1}^r \max_j\{|g_{i,j}(x)|^{e_i}\} \qquad (11)\\ &\le C' \cdot \max_{j_1, \dots, j_r}\left\{\left|\prod_{i=1}^r g_{i,j_i}(x)^{e_i}\right|\right\}.\end{aligned}$$

Thus we need to understand the polynomials

$$\prod_{i=1}^r g_{i,j_i}(x)^{e_i}.$$

By (7) we conclude that

$$x_0^{e_{r+1}+\cdots+e_s} \prod_{i=1}^r g_{i,j_i}(x)^{e_i} \in (\overline{f}_1, \dots, \overline{f}_k). \qquad (12)$$

Since the degree of g_{i,j_i} is at most z_i, by (7) the degree of the left-hand side in (12) is at most $\overline{B}(n, d_1, \dots, d_k)$. Thus there are polynomials $G_{i,j_1,\dots,j_r} \in K[x_1, \dots, x_n]$ of degree at most $\overline{B}(n, d_1, \dots, d_k) - d_i$ such that

$$\prod_{i=1}^{r} g_{i,j_i}(x)^{e_i} = \sum_{i=1}^{k} G_{i,j_1,\dots,j_r} f_i.$$

Note that if $h \in K[x_1, \dots, x_n]$ has degree at most q then there is a constant C'' such that

$$|h(x)| \le C'' \cdot (1 + ||x||)^q.$$

Thus for a suitable constant C''

$$|G_{i,j_1,\dots,j_r}(x)| \le C'' \cdot (1 + ||x||)^{\overline{B}(n,d_1,\dots,d_k)-d_i},$$

where C'' is independent of $i, j_1, \dots, j_r$. Therefore by (11)

$$\text{(13)} \qquad \begin{aligned} \operatorname{dist}(x, V)^{z_1e_1+\cdots+z_re_r} &\le C' \cdot \max_{j_1,\dots,j_r} \left\{ \left| \sum_i G_{i,j_1,\dots,j_r} f_i \right| \right\} \\ &\le kC'C'' \cdot \max_i\{|f_i(x)|\} \cdot (1 + ||x||)^{\overline{B}(n,d_1,\dots,d_k)}. \end{aligned}$$

Take $m = \sum_{i=1}^{r} z_i e_i$ and $C = kC'C''$ to get (5). □

Note that we proved in fact a slightly stronger statement:

$$\text{(14)} \qquad \operatorname{dist}(x, V)^m \le C \cdot \max_i\{|f_i(x)| \cdot (1 + ||x||)^{\overline{B}(n,d_1,\dots,d_k)-d_i}\}.$$

In this form both the bound on m and the exponents of $1 + ||x||$ are the best possible, provided that $d_i \ne 2$.

15. **Example.** This variant of an example given by Masser and Philippon (see [B1; K, 2.3]) shows that the upper bound on m in (14) is sharp (for $d_i \ne 2$).

Let $f_1 = x_2 - x_1^{d_1}$, $f_2 = x_3 - x_2^{d_2}$, $\dots$, $f_{n-1} = x_n - x_{n-1}^{d_{n-1}}$, $f_n = x_n^{d_n}$. Then $V(f_1, \dots, f_n) = \{0\}$. Let

$$x(t) = (t, t^{d_1}, t^{d_1 d_2}, \dots, t^{d_1 d_2 \cdots d_{n-1}}).$$

Then $\operatorname{dist}(x(t), 0) \approx |t|$ for small $|t|$ but

$$\max |f_i(x(t))| = |f_n(x(t))| = |t|^{d_1 d_2 \cdots d_n}.$$

16. **Example.** This example shows that in some cases the only value of m that works in (5) is $m = 1$.

In $K[x, y]$ let $f_1 = y$, $f_2 = y(x-1)^s - x$ where $s \ge 2$. Then $\overline{B}(2, 1, s+1) = s+1$ and $V(f_1, f_2) = \{0\}$. Consider the family of points $z(t) = (t, t(t-1)^{-s})$. Then $f_2(z(t)) = 0$ and for large values of $|t|$ we have

$$\operatorname{dist}(z(t), \{0\}) = ||z(t)|| \approx |t| \quad \text{and} \quad |f_1(z(t))| \approx |t|^{1-s}.$$

Thus

$$\max_{i=1,2}\{|f_i(z(t))| \cdot (1 + ||z(t)||)^{\overline{B}(n,d_1,d_2)-d_i}\} = |f_1(z(t))| \cdot (1 + ||z(t)||)^s \approx |t|.$$

Hence we must take $m = 1$ in (5).

This example also shows that the exponent 2 of $||z||$ in (6.2) cannot be made smaller if the degrees go to infinity.

17. *Remark.* One can interpret (5) as follows: Given a system of equations $f_1 = \cdots = f_k = 0$ over an algebraically closed field K, let $(x_1, \dots, x_n)$ be an approximate solution. Then there is an actual solution near $(x_1, \dots, x_n)$. From this point of view the assumption that V be nonempty is very inconvenient. This form is especially interesting when the absolute value is non-archimedean, e.g. when K is the algebraic closure of a complete discrete valuation ring. However in this case one would like to prove a similar result without assuming that K is algebraically closed, or even for equations over any complete local ring. Such results are known [A, Chapter 6] but the bounds are probably far from being optimal.

18. *Remark.* Let us take this opportunity to correct an error in [K]. In the formulation of [K, Proposition 1.10] the exponent should be the above $B(n, d_1, \dots, d_k)$ instead of $N(n, d_1, \dots, d_k)$. (See [K, Section 4].) These two functions are conjecturally equal but the equality is proved only if all the d_i are different from 2 [K, 1.9].

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